Math 1320: Solving Equations Reference Sheet

What is an equation? Previously, we learned to evaluate and simplify expressions. But now, instead of one expression, we'll be working with two equivalent expressions! We'll solve different types of equations: mathematical sentences where two expressions are equal. In solving these statements, we will find the value of our unknown (x) using the rules and properties we have learned when working with expressions.

Before, we worked with word problems that only involved solving linear equations (where the highest degree of any term in the equation equalled one). But now, we will learn to solve equations of different degrees and forms, including equations with polynomials, radicals, rational exponents, and equations in quadratic form.

Different Types of Equations

1. **Polynomial Equations**

$$3x^3 + 2x^2 = 12x + 8$$

A polynomial equation is the result of setting two polynomials equal to each other. The degree of a polynomial equation is equal to the highest degree of any term in the equation. When a polynomial equation is degree 1, the equation is linear. A polynomial equation of degree 2 is quadratic.

2. **Radical Equations**

$$\sqrt{2x + 15} - 6 = x$$

A radical equation is an equation where the variable is under a root.

3. **Equations with Rational Exponents**

$$8x^{5/3} - 24 = 0$$

Equations of this kind are when one or more term of the equation has a fractional exponent. Expressions with rational exponents can be represented with radicals.

4. **Equations That Are Quadratic in Form**

$$x^{\frac{2}{3}} + 6x^{\frac{1}{3}} + 9 = 0$$

Equations that are quadratic in form, are not polynomial equations of degree 2, but they may be written as quadratic equations using substitution. Equations that are quadratic in form contain an expression to a power, the same expression to that power squared, and a constant term. In the example above, the expression is $x^{1/3}$.

$\star$ Note: An equation that looks rational may actually be quadratic in form. Be sure to check before proceeding with either method.

Example 1. Polynomial Equations

$$3x^3 + 2x^2 = 12x + 8$$

A polynomial equation, like the example above, is created with two polynomials that are set equal to each other. When solving polynomial equations of degree 3 or higher, it may be helpful to rewrite it in standard form. Standard form is when one side of the equation is zero and the polynomial on the other side is written such that the terms are organized in descending powers of the variable.

Strategies for Solving Polynomial Equations		**Example**
Step 1	Move all terms to one side and obtain zero on the other side.	$3x^3 + 2x^2 = 12x + 8$ Subtract $12x$ and 8 from both sides: $3x^3 + 2x^2 - 12x = 8$ $3x^3 + 2x^2 - 12x - 8 = 0$
Step 2	Factor.	$3x^3 + 2x^2 - 12x - 8 = 0$ We have 4 terms in the polynomial, let's factor by grouping: $(3x^3 + 2x^2) + (-12x - 8) = 0$ $x^2(3x + 2) - 4(3x + 2) = 0$ $(x^2 - 4)(3x + 2) = 0$
Step 3	Set each factor equal to zero and solve the resulting equations.	$x^2 - 4 = 0$ $3x + 2 = 0$ $x^2 = 4$ $3x = -2$ $x = \pm\sqrt{4}$ $x = -\frac{2}{3}$ $x = 2,\ x = -2$
Step 4	Check all solutions in the original equation.	$x = 2$: $3x^3 + 2x^2 = 12x + 8$ $3(2)^3 + 2(2)^2 = 12(2) + 8$ $32 = 32$ $x = -2$: $3x^3 + 2x^2 = 12x + 8$ $3(-2)^3 + 2(-2)^2 = 12(-2) + 8$ $-16 = -16$ $x = -\frac{2}{3}$: $3x^3 + 2x^2 = 12x + 8$ $3\left(\frac{-2}{3}\right)^3 + 2\left(\frac{-2}{3}\right)^2 = 12\left(\frac{-2}{3}\right) + 8$ $0 = 0$

Example 2. Radical Equations

$$\sqrt{2x+15}-6=x$$

A radical equation is an equation with the variable under a root. In the example above, x is under a square root. When working with radical equations it is important to isolate the term with the radical. In other words, we need to get the radical term by itself on one side of the equation. This way we can 'undo' the radical by raising both sides to the power corresponding to the root. In the example above, to undo the square root, we will have to raise both sides of the equation to the power of two.

In solving radical equations, we may get extraneous solutions. These are solutions that do not make the original equation true. This is why it is important to check all solutions when solving equations.

Strategies for Solving Radical Equations		Example
Step 1	Isolate the radical on one side of the equation.	$\sqrt{2x+15}-6=x$ To get $\sqrt{2x+15}$ by itself, we need to add 6 to both sides: $\sqrt{2x+15}=x+6$
Step 2	Raise both sides to the power that 'undoes' the root.	$\sqrt{2x+15}=x+6$ Because we have a square root in the equation, we need to square both sides: $\left(\sqrt{2x+15}\right)^2=(x+6)^2$ $2x+15=(x+6)(x+6)$ $2x+15=x^2+12x+36$
Step 3	Solve the resulting equation.	$2x+15=x^2+12x+36$ Now, we have a polynomial equation. We can use the steps from the previous example to solve our equation: Move all terms to one side: $0=x^2+12x+36-2x-15$ Put in standard form: $0=x^2+10x+21$ Factor the trinomial: $0=(x+7)(x+3)$ Set each factor equal to zero and solve for x : $x+7=0$ $x+3=0$ $x=-7$ $x=-3$
Step 4	Check all solutions in the original equation.	$x=-7$: $\sqrt{2x+15}-6=x$ $\sqrt{2(-7)+15}-6=-7$ $\sqrt{1}-6=-7$ $1-6=-7$ $-5=-7$ FALSE $x=-3$: $\sqrt{2x+15}-6=x$ $\sqrt{2(-3)+15}-6=-3$ $\sqrt{9}-6=-3$ $3-6=-3$ $-3=-3$ TRUE

Since $x=-7$ makes our original equation false, we only have the solution $x=-3$.

Example 3. Equations with Rational Exponents

$$8x^{5/3} - 24 = 0$$

When we were introduced to exponent rules, we learned that a rational (fractional) exponent could be written as a radical and vice versa. For example, the equation above could be rewritten as:

$$8x^{5/3} - 24 = 0 \quad \Leftrightarrow \quad \sqrt[3]{8x^5} - 24 = 0$$

So, when we solve equations with rational exponents, we will follow steps similar to when solving radical equations. We will get the term with the rational exponent alone on one side, then raise both sides of the equation to a power that is the reciprocal (denominator and numerator of original fraction are switched) of the exponent.

An important thing to consider is if the numerator of the rational exponent is odd or even.

Numerator is Even	**Numerator is Odd**
$x^{\frac{m}{n}} = a$	$x^{\frac{m}{n}} = a$
$(x^{\frac{m}{n}})^{\frac{n}{m}} = \pm a^{\frac{n}{m}}$	$(x^{\frac{m}{n}})^{\frac{n}{m}} = a^{\frac{n}{m}}$
$x = \pm a^{\frac{n}{m}}$	$x = a^{\frac{n}{m}}$

We know this to be the case, since $(2)^2 = 4$ and $(-2)^2 = 4$, but $(2)^3 = 8$ and $(-2)^3 = -8$. An odd index has only one root. In the example above, the rational exponent is $\frac{5}{3}$ with a numerator of 5, therefore we will have only one solution.

Strategies for Solving Equations with Rational Exponents		**Example**
Step 1	Isolate the term with a rational exponent on one side of the equation.	$8x^{\frac{5}{3}} - 24 = 0$ To get $x^{\frac{5}{3}}$, we need to add 24 to both sides: $8x^{\frac{5}{3}} = 24$ Now, we divide both sides by 8: $x^{\frac{5}{3}} = 3$
Step 2	Determine if the numerator of the rational exponent is odd or even, then raise both sides of the equation to the power of the reciprocal.	$\frac{5}{3}$ has an odd numerator, so we will have only one solution. Now to get x by itself, we raise both sides to the $\frac{3}{5}$ power: $\left(x^{\frac{5}{3}}\right)^{\frac{3}{5}} = (3)^{\frac{3}{5}}$ $x = 3^{\frac{3}{5}}$ Rewrite the $3^{\frac{3}{5}}$ using radicals: $x = \sqrt[5]{3^3}$ Simplify: $x = \sqrt[5]{27}$
Step 3	Check all solutions in the original equation.	$8x^{\frac{5}{3}} - 24 = 0$ $8\left(\sqrt[5]{27}\right)^{\frac{5}{3}} - 24 = 0$ $8\left(\sqrt[5]{3^3}\right)^{\frac{5}{3}} - 24 = 0$ $8\left(3^{\frac{3}{5}}\right)^{\frac{5}{3}} - 24 = 0$ $8(3) - 24 = 0$ $24 - 24 = 0$ $0 = 0$ TRUE!

Example 4. Equations That Are Quadratic in Form

We know, from previous lessons, that a quadratic equation is an equation where the highest degree term is two. So, for an equation to be quadratic in form, it means the equation can be written as a quadratic equation by using substitution. Consider the two equations below:

$$x^{\frac{2}{3}} + 6x^{\frac{1}{3}} + 9 = 0 \qquad\qquad 3x^3 + 7x^2 - 1 = 0$$

The equation on the left could be written as $(x^{\frac{1}{3}})^2 + 6x^{\frac{1}{3}} + 9 = 0$ using exponent rules. Then, we could let $u = x^{\frac{1}{3}}$ to get the new quadratic equation $(u)^2 + 6(u) + 9 = 0$. But if we consider the equation on the right, can this be written as a quadratic equation with substitution? No. Using exponent rules, there is no way to write $3x^3$ as a power of 2. This equation is not quadratic in form.

Determine if the equations below are quadratic in form. If so, what substitution would you make?

1. $x^4 + 3x^2 - 4 = 0$ [Yes! Substitute $u = x^2$, since $(x^2)^2 = x^4$]
2. $3x^5 + 7x + 9 = 0$ [No. We cannot write $x^5 = (\ )^2$]
3. $x^{\frac{1}{2}} - 2x^{\frac{1}{4}} + 4 = 0$ [Yes! Substitute $u = x^{\frac{1}{4}}$, since $(x^{\frac{1}{4}})^2 = x^{\frac{1}{2}}$]

Once the given equation is converted to a quadratic form, we can solve our new equation similar to how we solved polynomial equations above.

Strategies for Solving Equations That Are Quadratic in Form		**Example**
Step 1	Determine if the equation is quadratic in form and, if so, what substitution to make.	$x^{\frac{2}{3}} + 6x^{\frac{1}{3}} + 9 = 0$ Notice that the equation has a term with $x^{\frac{1}{3}}$ and $x^{\frac{2}{3}}$ can be written as $\left(x^{\frac{1}{3}}\right)^2$. So, the equation is quadratic in form. Let $u = x^{\frac{1}{3}}$.
Step 2	Write the given equation as a quadratic equation with u. Solve for u.	$x^{\frac{2}{3}} + 6x^{\frac{1}{3}} + 9 = 0$ $\left(x^{\frac{1}{3}}\right)^2 + 6\left(x^{\frac{1}{3}}\right) + 9 = 0$ With $u = x^{\frac{1}{3}}$, we now have: $u^2 + 6u + 9 = 0$ Solve the polynomial equation: $u^2 + 6u + 9 = 0$ $(u+3)^2 = 0$ Factor. Use $(a+b)^2 = a^2 + 2ab + b^2$ $u + 3 = 0$ Set factors equal to zero $u = -3$ Solve for u
Step 3	Substitute the expression with x, back in for u. Solve for x.	$u = -3$ We let $u = x^{\frac{1}{3}}$, now: $x^{\frac{1}{3}} = -3$ Solve for x: $x^{\frac{1}{3}} = -3$ $\left(x^{\frac{1}{3}}\right)^3 = (-3)^3$ Raise each side to the power of $\frac{3}{1} = 3$ $x = -27$ Simplify
Step 4	Check all solutions in the original equation.	$x^{\frac{2}{3}} + 6x^{\frac{1}{3}} + 9 = 0$ $(-27)^{\frac{2}{3}} + 6(-27)^{\frac{1}{3}} + 9 = 0$ $9 + 6(-3) + 9 = 0$ $9 - 18 + 9 = 0$ $0 = 0$ TRUE!